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Bethe ansatz equations for general orbifolds of $\mathcal{N} = 4$ SYM

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Abstract: We consider the Bethe Ansatz Equations for orbifolds of $\mathcal{N}=4$ SYM w.r.t. an arbitrary discrete group. Techniques used for the Abelian orbifolds can be extended to the generic non-Abelian case with minor modifications. We also illustrate the interplay with the quiver gauge theory notation.

Keywords: 1/N Expansion, AdS-CFT Correspondence, Bethe Ansatz.

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1. Introduction

For a long time in high energy physics there exists a strong interest in the web of dualities between gauge theory and closed strings. The relationship essentially started with the inception of string theory as a dual model of hadronic interactions (for a recent review see, e.g., [1]). From the point of view of QCD, the modern theory of strong interactions, hadronic strings would be interpreted as color electric flux tubes between quarks. A concrete version of gauge/string correspondence was proposed by 't Hooft in [2] (see also [3 – 5]) in the form of the $1/N$ -expansion, the central idea of which is that the Feynman graphs of a large N gauge theory naturally organize themselves as triangulations of a string surface. The rank of the gauge group N is related to the string coupling via $g_S = 1/N$, and counts the number of handles of the surface spanned by the non-planar graphs. The gauge theory/closed string duality is expected to be a limit of a more general open/closed string correspondence, which should hold at the world sheet level. Open string diagrams are equivalent to closed string world sheets with holes. The idea behind the open/closed string correspondence is that the holes can be replaced by closed string vertex operators, and absorbed into an adjustment of the sigma model that governs the motion of the closed string. From the perspective of the low energy effective field theory, this relation between open and closed strings gives rise to the famous duality between gauge theory and gravity, the central example of which is the celebrated AdS/CFT correspondence [6 − 9]. The key physical insight that spurned this development was the discovery of the D-branes [10], followed by understanding of the geometrical nature of the non-Abelian Chan-Paton factors in terms of stacks of coincident branes [11]. On the other side, in terms of the Matrix Theory proposal [12] non-Abelian gauge degrees of freedom are just a part of a more general theory; and thus they naturally incorporate into the web of dualities.

However, there are some difficulties in studying the AdS/CFT conjecture. One of them is the fact that the weak coupling on the gravity side (closed strings) corresponds to the strong coupling regime on the gauge theory side (open strings); and this prevents one from performing simple perturbative checks. It was major breakthrough when it was realized that some integrable structures were present in the scalar subsector of $\mathcal{N} = 4$ SYM [13], and this result was extended to the complete set of operators in [14 − 16]. At the same time there was investigated the integrability of the closed string motion in [17] and the following works. This opened the new opportunities of understanding the AdS/CFT duality beyond perturbation theory.

Another idea commonly used in string theory since [18] is that of the orbifold space. An orbifold is a quotient of some manifold w.r.t. a discrete group. The procedure of orbifoldization was expected to be useful in particular for model building. A strong motivation for this is the usage of quotient spaces for (super)symmetry breaking. Another way to use orbifold construction which was used recently is to embed some models into quiver gauge theories.

Even though there are some works studying these dualities for some special orbifolds or some special limits [19 − 27]; they mainly deal with the Abelian orbifolds and the corresponding quiver gauge theories. The main goal of this paper is to extend some of these studies to the generic orbifolds with an arbitrary non-Abelian orbifold group. Organization of the paper is as follows. In the second section we introduce the orbifold gauge theory which is the low-energy limit of the corresponding open string theory. We discuss the subtleties specific to the general non-Abelian orbifolds. We introduce the two different descriptions, the one using the twist fields (and most closely resembling the original unorbifolded theory) as well as the one using the quiver diagram. Then we develop the transition formulae between them. From the construction of observables it becomes clear that in a given twisted sector one can diagonalize the twist field, and it allows one to apply the techniques used for the Abelian orbifolds in the general case. In the third section we review the Bethe Ansatz Equations (BAE) and generalize them to the generic orbifold theories. The key ingredients of the construction are essentially the same as those for the Abelian orbifolds. The key idea is the mentioned diagonalization of the twist field in a given twisted sector, after which the setup reduces to the Abelian case modulo some subtleties. In the fourth section we study some applications of the BAE. We consider particular quivers (both Abelian and non-Abelian) and show how the eigenvectors of the matrix of anomalous dimensions are rewritten in terms of the quiver notation. Appendix

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A contains the calculations related to the conversion between the two descriptions in the orbifold gauge theory as well as the construction of observables. Appendix B summarizes the facts about the simplest non-Abelian group D_n.

2. Orbifold gauge theory

We now turn to the study of the class of quiver gauge theories obtained by taking an arbitrary (Abelian or non-Abelian) orbifold of $\mathcal{N}=4$ supersymmetric $\mathrm{U}(N)$ gauge theory. Our motivation is to investigate to what extent the recently uncovered large N integrability of $\mathcal{N}=4$ SYM can be extended to this general class of orbifold gauge theories. In this section we will summarize some of their relevant properties. The relevant references are [28 – 33].

It will be convenient to think of the quiver gauge theory as the low energy limit of the open string theory on a stack of N D3-branes located near an orbifold singularity. Before taking the orbifold quotient, the transverse space of the D3-branes is $\mathbb{R}^6 \simeq \mathbb{C}^3$. The low energy field theory on the D3-branes is $\mathcal{N}=4$ SYM, with its field content (in $\mathcal{N}=1$ superfield notation): a vector multiplet $\mathcal{A}$ and three chiral multiplets Φ^I, with $I=1,2,3$, that parameterize the three complex transverse positions of the D3-branes along $\mathbb{C}^3$.

Let Γ be some finite group of order $|\Gamma|$, that acts on $\mathbb{C}^3$. The orbifold space is obtained by dividing out the action of the discrete group Γ. The transverse space to the D3-branes thus becomes $\mathbb{C}^3/\Gamma$. When viewed from the covering space, the stack of N D3-branes in the orbifold space give rise to the total of $|\Gamma|N$ image D3-branes. It is convenient to label the image D3-branes by a pair of Chan-Paton indices (i,h) with $i=1,\dots,N$ and $h\in\Gamma$, so that the brane labeled by (i,h) is the image of the i-th brane inside the D3-stack under the group element $h\in\Gamma$. The group Γ thus acts on the Chan-Paton indices as

$$g \;:\; (i,h) \;\rightarrow\; (i,gh)\,. \tag{2.1}$$

When the N coincident D-branes all approach the orbifold fixed point, the image branes all coincide and the strings stretched between them have massless ground states. The vector multiplet $\mathcal{A}$ has a separate matrix entry for each open string stretching between two image branes, and thus defines an $|\Gamma|N\times|\Gamma|N$ matrix. Before imposing invariance under the orbifold group Γ, the full collection of image branes thus supports an $\mathrm{U}(|\Gamma|N)$ gauge symmetry. The orbifold projection, however, selects only those fields that are invariant under the discrete group Γ. The discrete group acts on the vector multiplet $\mathcal{A}$ only via the Chan-Paton indices and on the chiral multiplets Φ^I via both the Chan-Paton and transverse indices.

This projection operator does not commute with the full $\mathcal{N}=4$ superconformal invariance, but in the special case that Γ forms a subgroup of SU(3), the orbifold quotient preserves $\mathcal{N}=1$ superconformal invariance. More generally, one could consider orbifolds with Γ some subgroup of SO(6). However, it has been shown that for non-supersymmetric orbifolds, the quantum theory has non-zero β-functions for certain double-trace operators and is therefore not conformally invariant. The renormalized Hamiltonian of non-supersymmetric orbifolds thus contains extra terms that do not descend from the $\mathcal{N}=4$ Hamiltonian [34]. For this reason we will restrict ourselves to the supersymmetric subclass.

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Although the orbifold theories all have less supersymmetry, their action is assumed to be identical to that of the parent $\mathcal{N}=4$ theory, which in $\mathcal{N}=1$ superfield notation reads

$$\mathcal{L} = \int d^4\theta \, \mathrm{Tr}\left(\mathcal{W}^\alpha \mathcal{W}_\alpha + e^{\mathcal{A}} \Phi_I^\dagger e^{-\mathcal{A}} \Phi_I\right) \; + \; \int d^2\theta \, \epsilon^{IJK} \, \mathrm{Tr}\left(\Phi_I [\Phi_J, \Phi_K]\right) \; + \; c.c. \qquad (2.2)$$

Here the trace Tr is over the adjoint representation of the full $\mathrm{U}(|\Gamma|N)$ gauge group of the $\mathcal{N}=4$ theory. The fields $(\mathcal{A}, \Phi)$ of the orbifold theory, however, have to be Γ-invariant. This invariance condition can be solved as

$$\mathcal{A}_{h,hg} = \mathcal{A}(g)\,, \qquad (2.3)$$

$$\Phi^I_{h,hg} = \mathfrak{R}(h)^I{}_J \phi^J(g)\,. \qquad (2.4)$$

We see that after the projection, the Γ valued left and right Chan-Paton indices have collapsed to a single group valued index. The gauge and matter fields can thus be thought of as group algebra valued $N \times N$ matrices. We will refer to the above basis of orbifold projected fields as the *orbit basis* (as distinguished from the *quiver basis*, that will be introduced later).

Note that the orbifold projection does not commute with the full $\mathrm{U}(|\Gamma|N)$, the gauge symmetry gets broken to a subgroup. This unbroken gauge group is identified as follows. Recall that the orbifold group acts on the Chan-Paton indices of the gauge field via the regular representation, and the latter decomposes into irreducible representations ρ_λ via

$$\gamma_{\mathrm{reg}}(g) = \bigoplus_\lambda \rho_\lambda(g)^{\oplus N_\lambda}\,, \qquad N_\lambda = \dim \rho_\lambda\,. \qquad (2.5)$$

In words, each irreducible representation ρ_λ occurs N_λ times in the decomposition of the regular representation. In explicit matrix notation, we have

$$\gamma_{\mathrm{reg}} = \begin{pmatrix} \rho_1 \otimes \mathbb{1}_{N_1} & 0 & \dots & 0 \\ 0 & \rho_2 \otimes \mathbb{1}_{N_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_r \otimes \mathbb{1}_{N_r} \end{pmatrix}\,, \qquad (2.6)$$

where each ρ_λ denotes an $N_\lambda \times N_\lambda$ matrix. By inspecting the explicit form (2.6) of γ_{reg}, it is not difficult to derive that the orbifold gauge group, defined as the subgroup of $\mathrm{U}(|\Gamma|N)$ transformations that commutes with $\gamma(g)$ for all $g \in \Gamma$, takes the product form

$$\bigotimes_\lambda \mathrm{U}(NN_\lambda)\,. \qquad (2.7)$$

Here the product runs over all representations of Γ and each factor $\mathrm{U}(NN_\lambda)$ is the subgroup that rearranges the NN_λ copies of the representation space V_λ of ρ_λ — it therefore obviously commutes with Γ. Using Schur's lemma, one proves that (2.7) indeed defines the maximal unbroken gauge group: physical operators need only be gauge invariant under this group.

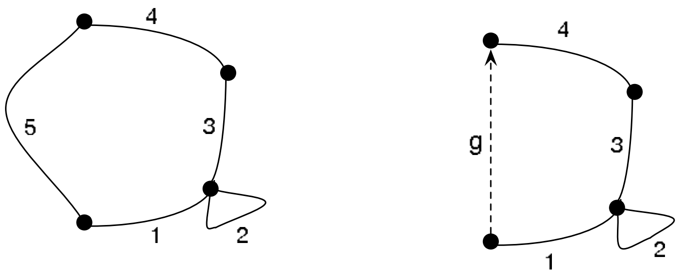

Figure 1: An untwisted state (left) and twisted state (right). Both are concatenated arrays of open strings (lines) stretched between D3-branes (dots). The end-point brane of the twisted state is the image under a finite group transformation g on its begin-point brane.

2.1 Construction of observables

The novel feature of the orbifold gauge theory is the emergence of the twisted states, which are new compared to the parent theory. The untwisted sector is the subclass of operators that come directly from the parent $\mathcal{N}=4$ theory. In the open string language, the untwisted operators can be thought of as arrays of concatenated open strings attached to several image D3-branes, as indicated on the left in the Fig 1. Such an operator is written as

$$\mathcal{O} = \mathrm{Tr}\left(\mathcal{W}_{A_1}\mathcal{W}_{A_2}\dots\mathcal{W}_{A_L}\right). \tag{2.8}$$

Here $\mathcal{W}_A$ stands for a (multiple) covariant derivative of one of the fields of the theory, in $\mathcal{N}=1$ notation:

$$\mathcal{W}_A \in \left\{ \mathcal{D}^n\Phi_I,\, \mathcal{D}^m\mathcal{W}_\alpha \right\}; \tag{2.9}$$

and each operator $\mathcal{W}_A$ corresponds to a ground state of one of the open strings. The gauge invariant trace implies that the array is closed: it begins and ends on the same D3-brane, and thus represents a proper closed string state in the unorbifolded theory.

A twisted sector state, on the other hand, corresponds to a concatenated array of open strings that ends on a different D3-brane, related via a finite group transformation $g \in \Gamma$ to the D3-brane where it begins. This configuration looks like an open string on the covering space, but it represents a closed string on the orbifold space. Correspondingly, it is associated with an operator that is not gauge invariant under the full $\mathrm{U}(|\Gamma|N)$ symmetry of the cover theory, but that *is* invariant under the gauge group (2.7) of the orbifold theory. In the gauge theory, the twisted states are represented as single trace expressions

$$\mathcal{O}(g) = \mathrm{Tr}\left(\gamma(g)\,\mathcal{W}_{A_1}\mathcal{W}_{A_1}\dots\mathcal{W}_{A_L}\right), \tag{2.10}$$

where we introduced a twist operator $\gamma(g)$, defined as follows. When $\gamma(g)$ acts from the left on a matrix-valued operator $\mathcal{W}_A$, it acts via the group action (2.1) — the regular representation $\gamma_{\mathrm{reg}}(g)$ — on the left Chan-Paton index. When $\gamma(g)$ acts from the right, it acts via the complex conjugate group action $\overline{\gamma}_{\mathrm{reg}}(g)$ on the right Chan-Paton index. The actions from the left and from the right are not identical; instead, the operators $\mathcal{W}_A$ of the orbifold theory satisfy a relation of the form

$$\gamma(g)\,\mathcal{W}_{A} \;=\; \mathfrak{R}(g^{-1})_{A}^{\;B}\,\mathcal{W}_{B}\gamma(g)\,, \tag{2.11}$$

where $\mathfrak{R}(g)_A^{\;B}$ denotes a matrix representation of the finite group Γ, acting on the $\mathbb{C}^3$ index of $\mathcal{W}_A$.[1]

As a consequence of the orbifold projection, some of the physical operators (2.10) vanish identically. Using equation (2.11) to commute $\gamma(g)$ past all the fields in the operator shows that a necessary condition for non-vanishing operators is that the total single trace operator must be invariant under the simultaneous action of $\mathfrak{R}(g)_A^{\;B}$ on all the spins. However, while necessary, this is not sufficient. More generally, physical operators are of the form

$$\mathcal{O}_{\mathcal{K}}(g) \;=\; \mathcal{K}^{A_1 A_2 \cdots A_L}\,\mathrm{Tr}\left(\gamma(g)\,\mathcal{W}_{A_1}\mathcal{W}_{A_1}\ldots\mathcal{W}_{A_L}\right), \tag{2.12}$$

where $\mathcal{K}$ must be an invariant tensor under the complete stabilizer subgroup S_g of g, defined as the subgroup within Γ of all elements that commute with g.[2] In the untwisted case, where g is the identity element in Γ, the stabilizer subgroup is the whole group Γ and indeed, as we saw before, untwisted operators are in one-to-one correspondence with Γ-invariant tensors. It is important to note that the basis (2.12) of operators is a complete basis, in the sense that any operator of the seemingly more general class given in (2.10), that is not of the form (2.12), vanishes identically. Detailed construction of twisted operators as well as the proof of their gauge invariance is given in appendix A.1. Another important fact is that the twist field $\gamma(g)$ suffices to produce all the operators within the twist class $[g]$ (cf. [35]). Then the untwisted operators can naturally be viewed as those belonging to the sector with the conjugacy class of the unit element $[e]$. Then any single representative $g \in [g]$ can be diagonalized; and then one can apply the tools used for the Abelian orbifolds to the general non-Abelian case. If g is an element of order S in Γ, then its eigenvalues are some phase factors of the form $\exp(2\pi i s_k/S)$. This observation will be exploited in section 3 where we study the Bethe Ansatz Equations for the orbifold gauge theory.

2.2 Feynman rules

As an example we go through the derivation of the Feynman rules for the scalar field ϕ^I; the other fields are treated in a similar way. We can parameterize the invariant configurations of the scalar field $\phi^I_{ig,jh}$ in terms of this group algebra valued object $\phi^I_{ij}(g)$, and this group algebra valued field ϕ is to be integrated over in the path integral. Using the parameterization (2.4) and the orthogonality of the defining representation $\mathfrak{R} : \Gamma \to \mathrm{SO}(6)$,

[1] Here $\mathfrak{R}(g) = 1$ in case $\mathcal{W}_A$ has no $\mathbb{C}^3$ index. Note further that inserting multiple twist operators in the trace does *not* introduce a new class of operators, since by using the exchange relation (2.11) and the property $\gamma(g_1)\gamma(g_2) = \gamma(g_1 g_2)$, one can always reduce any number of twist operators to a single overall twist. This is as one would have expected from the string interpretation.

[2] It can be the case that even for some S_g-invariant tensor $\mathcal{K}(g)$ the corresponding operator $\mathcal{O}_{\mathcal{K}}$ vanishes identically due to some symmetry reasons — for instance, this is the case in the $\mathbb{Z}_6$ quiver we consider in section 4.

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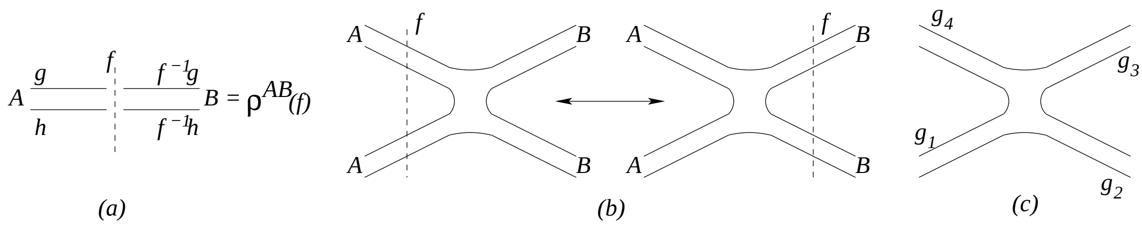

Figure 2: (a) When a line of the Feynman graph crosses the cut the Wick contraction $\langle \mathcal{W}_A\, \mathcal{W}_B \rangle$ is non-diagonal, and proportional to the matrix element $\mathfrak{R}^{AB}(f)$. (b) The twist lines can be deformed and moved through interaction vertices. (c) In the notation using the group valued fields untwisted vertices obey the conservation condition, similar to the conservation of momentum: for the vertex shown the product $g_1 g_2 g_3 g_4 = 1$.

we get the kinetic term for the scalar field in the form

$$\mathcal{L}_{\phi\phi} = \frac{1}{2}\sum_I \operatorname{Tr} \partial_\mu \phi^I\, \partial^\mu \phi^I = \frac{1}{2}|\Gamma| \sum_g \sum_{I,J} \mathfrak{R}(g)^I_J\, \partial_\mu \operatorname{Tr} \phi^I(g)\, \partial^\mu \phi^J(g^{-1})\,. \tag{2.13}$$

Then for the quadratic propagator the only modification compared to the original theory is "conservation of the group index" and renormalization:

$$\left\langle \phi^I_{ij,g}\, \phi^J_{kl,h} \right\rangle = |\Gamma|^{-1} \frac{\mathfrak{R}(g)^I_J}{p^2}\, \delta_{gh,e}\, \delta_{il}\, \delta_{jk}\,, \tag{2.14}$$

In terms of the original $\mathcal{N}=4$ fields (we omit the obvious Latin part of the Chan-Paton indices)

$$\left\langle \phi^I_{h,g}\, \phi^J_{f^{-1}g,\, f^{-1}h} \right\rangle = \frac{\mathfrak{R}(f)^I_J}{|\Gamma|\, p^2}\,. \tag{2.15}$$

Generally, for elementary fields (or their derivatives) $\mathcal{W}_A$ there takes place the following replacement in the propagator:[3]

$$\langle \mathcal{W}_A\, \mathcal{W}_B \rangle_{\mathcal{N}=4} = G(p)\, \delta^{AB} \;\to\; \langle \mathcal{W}_A\, \mathcal{W}_B \rangle = \frac{1}{|\Gamma|}\, G(p)\, \mathfrak{R}^{AB}(f)\,. \tag{2.16}$$

The factor of $1/|\Gamma|$ compensates for the overcounting of fields.

The propagator is not simply diagonal on the group valued Chan-Paton indices (g, h), but there can be a twist by some group element f, that acts simultaneously on both the left and right index. The advantage of this (redundant) double line notation is that the interaction vertices coincide with those of the original theory, and the only modification is the introduction of these twists along the propagators.

Equivalently, we can think of the twist as the assignment of a group element f to each line of the dual graph to the Feynman diagram. We will call these lines on the dual

[3]We ignore the ghost fields. Gauge fixing is easy to do via the Feynman gauge. Since the gauge field $\mathcal{A}$ can be treated as a group algebra valued, the gauge fixing and Faddeev-Popov ghosts can also be treated as group algebra valued.

graph 'cuts'. When a propagator crosses a cut, the propagator $\langle \phi^I \, \phi^J \rangle$ is non-diagonal: the conventional factor δ^{IJ} gets replaced by the matrix element $\mathfrak{R}^{IJ}(f^{-1})$ with f the twist along the cut. Vertices of the dual graph correspond to loops of the original Feynman graph. The product of the group elements that meet at a dual vertex must multiply to the identity element in Γ. (Unless the amplitude involves the insertion of some twist operator at this dual vertex, see below.)

2.3 Quiver gauge theory

In this section, we will make a comparison between the above group theoretic description of the physical operators with the quiver representation of the orbifold gauge theory. It is the quiver gauge theory representation that makes the physical field content of orbifold gauge theories most manifest. As discussed, the unbroken gauge group of the orbifold theory takes the product form

$$\bigotimes_{\lambda} \mathrm{U}(NN_\lambda)\,, \tag{2.17}$$

where the product runs over all representations ρ_λ of the finite group Γ, and $N_\lambda = \mathrm{dim}V_\lambda$. Notice that, even in the case that $N=1$, i.e, for the world-brane theory of a single D3-brane near an orbifold singularity, this gauge group contains several, in general non-Abelian, factors. In the string theoretic construction, each gauge factor is associated to a stack on NN_λ so-called fractional D3-branes. There is one type of fractional brane for each representation ρ_λ of the finite group.

The vector multiplets $\mathcal{A}$ arise as the ground states of open strings attached to a given fractional brane. Let us denote by $\mathcal{A}_\lambda$ the vector multiplet of the fractional brane associated to ρ_λ. Hence $\mathcal{A}_\lambda$ is an $\mathrm{U}(NN_\lambda)$ gauge multiplet. In terms of the *orbit basis* $\mathcal{A}(g)$ defined in (2.3), the *quiver basis* $\mathcal{A}_\lambda$ is obtained via the Fourier-like transformation (see appendix A):

$$\mathcal{A}_\lambda = \sum_g \, \bar{\rho}_\lambda(g)\, \mathcal{A}(g) \tag{2.18}$$

Setting up the quiver terminology, we will refer to each gauge factor and its associated stack of fractional branes, as a *node* of the quiver diagram. There is one quiver node for each irreducible representation of Γ.

In a quiver diagram, the nodes are connected by oriented lines: these represent the chiral matter fields. In the string theory construction, the chiral matter fields Φ^I arise as the ground states of open strings that may have end-points on two different fractional branes. Correspondingly, they transform as bi-fundamental fields under the product gauge group (2.17). Algebraically the chiral matter fields $\Phi^{\lambda\overline{\mu}}$ correspond to the invariant tensors $(\mathbb{C}^3 \otimes V_\lambda \otimes \overline{V}_\mu)^\Gamma$. The number $n_{\lambda\bar{\mu}}$ of chiral matter fields between two given nodes λ and μ is determined by the multiplicity of ρ_μ in the decomposition of the tensor product between the defining representation $\mathfrak{R}$ and ρ_λ:

$$\mathfrak{R} \otimes \, \rho_\lambda = \bigoplus_\mu \, n_{\lambda\bar{\mu}} \, \rho_\mu \,. \tag{2.19}$$

In the string construction, the number $n_{\lambda\bar{\mu}}$ is the intersection number between the two fractional branes. The fields $\Phi^{\lambda\bar{\mu}}$ thus transform in the $(NN_\lambda, \overline{NN_\mu})$ bi-fundamental representation of the gauge group (2.17).[4] This *quiver basis* $\Phi^{\lambda\bar{\mu}}$ is related to the *orbit basis* $\Phi^I(g)$ given in (2.4) via the linear transformation

$$\Phi^{\lambda\bar{\mu}} = \sum_{g,I} \mathcal{K}^{\,I}_{\lambda\bar{\mu}}\, \bar{\rho}_\mu(g)\, \Phi_I(g)\,, \tag{2.20}$$

where $\mathcal{K}_{\lambda\bar{\mu}}$ denotes one of the $n_{\lambda\bar{\mu}}$ basis elements that spans the space of invariant tensors in $\mathbb{C}^3 \otimes V_\lambda \otimes \overline{V}_\mu$.

In the quiver basis, it is now easy to specify all possible single trace operators of the orbifold gauge theory. For this, it is useful to introduce the notion of the *path algebra* of the quiver diagram. A path is a concatenated array of arrows that connect quiver nodes connected by oriented lines. The arrows are allowed to point back to the same node. We can multiply two paths if one ends at the same node as where the other begins. We can then connect them head to tail to produce a single longer path. In the quiver gauge theory, each arrow of the path represents a gauge or matter operator $\mathcal{W}_A$ of the general form (2.9), transforming in the corresponding representation of the quiver gauge group. Connecting two arrows amounts to taking their gauge invariant product at the corresponding quiver node. To write gauge invariant operators, we now simply choose arbitrary closed paths along the quiver, pick a corresponding array of operators, and in the end take the trace.

How does this description of gauge invariant single trace operators compare with that in terms of twisted sector states (2.12)? Let us pick some closed path $\mathcal{C}_\lambda$, that starts and ends at a given node λ but along the way visits the following sequence of quiver nodes

$$\mathcal{C}_\lambda \ \ : \ \ \lambda \,\leftarrow\, \mu \,\leftarrow\, \nu \,\leftarrow \ldots \,\leftarrow\, \sigma \,\leftarrow\, \lambda\,. \tag{2.21}$$

For each arrow along this path, we pick the corresponding field and multiply them together, and take the trace at the λ node

$$\mathcal{O}_{\mathcal{C}_\lambda} = \mathrm{Tr}_\lambda\big(\mathcal{W}_{\lambda\bar{\mu}}\mathcal{W}_{\mu\bar{\nu}}\cdots\mathcal{W}_{\sigma\bar{\lambda}}\big)\,. \tag{2.22}$$

This is a manifestly gauge invariant operator of the quiver gauge theory. The equivalence with the group algebraic description of the orbifold theory implies that this operator must be a linear combination of twisted state operators $\mathcal{O}_\mathcal{K}(g)$ defined in eq. (2.12). A straightforward calculation, described in appendix A, indeed shows that

$$\mathcal{O}_{\mathcal{C}_\lambda} = \sum_g \mathcal{K}(g)^{A_1A_2\ldots A_L}\, \mathrm{Tr}\left(\gamma(g)\,\mathcal{W}_{A_1}\ldots\mathcal{W}_{A_L}\right), \tag{2.23}$$

where the S_g-invariant tensor $\mathcal{K}(g)$ is given by

$$\mathcal{K}(g)^{A_1A_2\ldots A_L} = \mathrm{Tr}_\lambda\big(\overline{\rho}_\lambda(g)\,\mathcal{K}^{A_1}_{\lambda\bar{\mu}}\mathcal{K}^{A_2}_{\mu\bar{\nu}}\cdots\mathcal{K}^{A_L}_{\sigma\bar{\lambda}}\big)\,. \tag{2.24}$$

The class of operators associated to closed loops on the quiver diagram span a complete basis of twisted sector operators, and vice versa.

[4] As a check, let us count the number of independent components of the chiral matter field Φ. For each arrow there are $N^2N_\lambda N_\mu$ components, and each node therefore connects to $N^2N_\lambda\sum_{\{\mu\}}\dim V_\mu$ independent components. Since $\mathfrak{R}\otimes V_\lambda = \oplus_{\{\mu\}}V_\mu$, dimension counting gives $\sum_{\{\mu\}}\dim V_\mu = 3\dim V_\lambda$. Therefore, the total number of independent components of Φ is $3N^2\sum_\lambda N_\lambda^2 = 3\,|G|\,N^2$. This is the expected result.

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3. Integrability: orbifolding the Bethe ansatz

The field theoretic problem we are trying to solve on the gauge theory side is diagonalization of the matrix of anomalous dimensions in the large N (planar diagram) limit. It is convenient to represent the field theory operators as the spin chain states,

$$\mathcal{O}^{A_1 A_2 \dots A_L}[g] \equiv \mathrm{Tr}\left(\gamma(g)\mathcal{W}_{A_1}\mathcal{W}_{A_2}\dots\mathcal{W}_{A_L}\right) = |A_1 A_2 \dots A_L\rangle_g \,. \tag{3.1}$$

Using this terminology, the matrix of anomalous dimensions is represented as some spin chain Hamiltonian $\mathcal{H}$. Note that the basis (3.1) is overcomplete — some of the states are projected out. Another subtlety is that one can perform a cyclic permutation in the trace leading to a seemingly different spin chain representation. In the untwisted case this results in an extra requirement on the physical spin chain state (i.e., one emerging from some gauge theory operator) — invariance w.r.t. the translation operator, the zero momentum condition. This particular choice of a representative makes the form of the Hamiltonian the most simple. When a non-trivial twist field $\gamma(g)$ is introduced, the zero momentum constraint gets modified.

Since all the terms in the action are untwisted, in the planar limit there should be no mixing between the sectors with different twists. A twisted sector is therefore a superselection sector: the twist $[g]$ is preserved under time-evolution defined by $\mathcal{H}$. This way we can restrict ourselves to the operators $\mathcal{O}[g]$ with a fixed class $[g]$. However, the representation of $\mathcal{H}$ as a spin chain Hamiltonian does depend on the twist sector. This dependence can be derived based on the form of the Feynman rules. The sum over the twist factors locally decouples from the remainder of the Feynman integral. In particular, the Γ-invariance of the interaction vertices of the original $\mathcal{N}=4$ Feynman diagram ensures that the cuts can be deformed and moved through the vertices, as it is indicated in figure 2. Following this procedure one can move the cuts, and translate them along the worldsheet spanned by the Feynman diagram. Evidently, we can merge cuts that are along homologous cycles on the worldsheet; the group element associated with the merged cut is the product of the original twists.[5] Proceeding this way, we can merge all the cuts and reduce the sum over the twist factors to a single set of twists associated to a generating set of non-contractible loops of the worldsheet spanned by the Feynman diagram.

Note that each operator insertion corresponds to a hole (puncture) on the graph surface, and a planar diagram that describes the leading order large N limit of amplitudes of some operators of the orbifold gauge theory (3.1), can be drawn on a cylinder (or a sphere with the two punctures). In the untwisted sector there is only one non-contractible loop wrapping the cylinder. Summation over this twist leads to projection onto the Γ-invariant states. Hence in this case, the amplitudes of the orbifold coincide with those of the $\mathcal{N}=4$ theory, as advocated. The miraculous integrability of the $\mathcal{N}=4$ theory therefore directly carries over to the untwisted sector of the orbifold gauge theory, provided it is supersymmetric.

[5] Summation over the different configurations leading to the same overall cut results in renormalization $N \to |\Gamma|\, N$ in the $1/N$ expansion; cf. [33].

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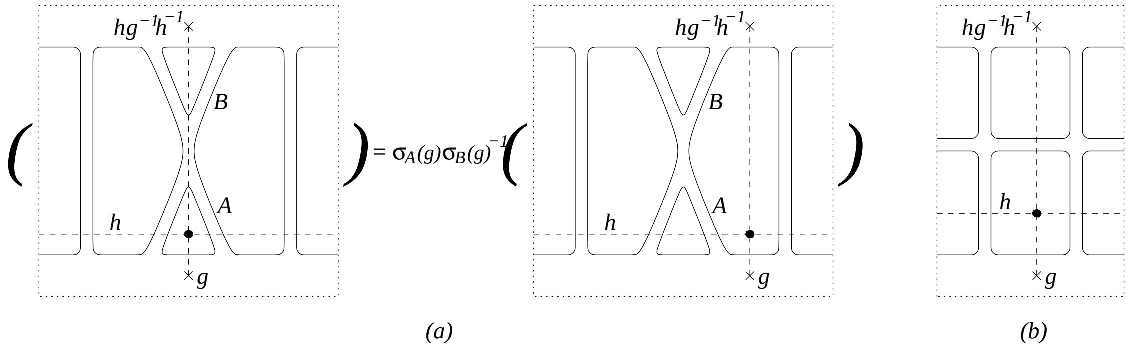

Figure 3: (a) A planar diagram on a cylinder. Introduction of the twist field causes appearance of an extra vertical cut in the dual graph (dotted lines). Should this cut be located in the interaction region it can be shifted away using the interchange relation (2.11). Representation matrix $\mathfrak{R}(g)$ being diagonalized, this shift results in a mere phase factor $\sigma_A(g)\sigma_B(g)^{-1}$. Note that the twist field gets conjugated, $g \to hgh^{-1}$, and this does not change its class. Then the summation over the cut h results in the projection onto the S_g-invariant subspace. (b) Diagrams with high number of loops can contain the wrapping interactions which do not reduce to the untwisted case. The diagram shown would be multiplied by an extra factor, the character $\operatorname{Tr}\mathfrak{R}(g)$ as a result of the horizontal loop wrapping the cylinder.

The story with the twisted sectors is slightly more complicated. In terms of the dual graph each twist field can be represented as a tadpole ending in the corresponding puncture. A direct consequence is the fact that the standard form of the dual graph consists of one horizontal cut wrapping the cylinder and one vertical cut corresponding to the twist (figure 3). However, the extra cut can be moved away from the interaction region using the commutation relation (2.11). After this transformation the graph coincides with that of the $\mathcal{N}=4$ theory modulo renormalization $N \to |\Gamma|\, N$ and projection onto the S_g-invariant states in (3.1). Unfortunately, this equivalence extends only up to the $\ell < L$ loops. The reason for this restriction is that the ℓ-loop gauge theory Hamiltonian translates into a semi-local spin chain Hamiltonian that connects $\ell + 1$ adjacent spins. So when $\ell \geq L$, the Hamiltonian becomes fully delocalized, and includes the so-called wrapping terms, non-local interactions that wrap around the full length of the spin chain. When this is the case, the extra cut emerging from insertion of the twist field $\gamma(g)$ can no longer be shifted away from the interaction region, and some propagators inevitably cross it (figure 3b).

The conclusion is that locally, on any nearest neighbor set of spins, the interaction terms in $\mathcal{H}$ all act identically to the local interaction terms of the $\mathcal{N} = 4$ Hamiltonian, as long as the local set of spins does not contain the twist operator $\gamma(g)$. If the twist generator is present in the interaction region, one could shift the twist operator to either side, until it is outside the interaction region. In this way we derive, for example, that the nearest neighbor interaction term, when acting on two spins separated by a twist $\gamma(g)$, gets modified via

$$\mathcal{H}_{[12]}\, \mathcal{W}_A \gamma(g)\, \mathcal{W}_B \;=\; \widetilde{\mathcal{H}}_{AB}^{CD}\; \mathcal{W}_C \gamma(g)\, \mathcal{W}_D\,, \tag{3.2}$$

where

$$\widetilde{\mathcal{H}}_{AB}^{CD} = \mathcal{H}_{AB'}^{CD'} \, \mathfrak{R}(g)_B^{B'} \, \overline{\mathfrak{R}}(g)_{D'}^{D} \,. \tag{3.3}$$

This relation (and analogous relations for the higher order terms) expresses the Γ-invariance of the local interaction terms of $\mathcal{H}$ — the twist field can be moved either to the left or to the right, which results in the same phase factor.

It is important that in each given twisted sector $[g]$ one can diagonalize the twist field $\gamma(g)$ and apply the methods that are used for the Abelian orbifolds to the general case.

3.1 Bethe equations: a brief introduction

We will start with the simplest example, the periodic Heisenberg $\mathfrak{su}_2$ spin chain of length L. Each of the L spins has a two-dimensional space of states $\mathbb{C}^2$ with the basis vectors $|\downarrow\rangle$ and $|\uparrow\rangle$ corresponding to the spin being oriented downward or upward. On the field theory side this picture corresponds to the $\mathfrak{su}_2$ subsector consisting of the two scalar fields Z and W. Our goal is to diagonalize the Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{L} \Big(1 - \mathbf{P}_{i,i+1} \Big) \,, \tag{3.4}$$

where $\mathbf{P}_{i,i+1}$ is the interchange operator acting between the i-th and the $i+1$-th sites. We choose a vacuum state $|\downarrow\downarrow\ldots\downarrow\rangle$ with all spins pointing down. ($\operatorname{Tr} Z^L$ operator in field theory.) The next step is to consider states with one excitation,

$$|n\rangle = |\downarrow\downarrow\ldots\downarrow\uparrow_n\downarrow\ldots\downarrow\rangle \tag{3.5}$$

with the spin up being at the n-th position. One can try to find a plane wave solution in the form

$$|k\rangle = \sum_{n=1}^{L} \mathrm{e}^{ikn} \, |n\rangle \,. \tag{3.6}$$

Acting with the Hamiltonian, we get for the eigenvalue $\epsilon(k) = 1 - \cos k$. One still has to identify $|0\rangle \equiv |L\rangle$, and this leads to the periodicity condition

$$\mathrm{e}^{ikL} = 1 \,. \tag{3.7}$$

Physically such a solution corresponds to a standing wave. The next step is to consider a solution with several waves. A remarkable feature of this system is its integrability. It manifests itself in the fact that the scattering reduces to the two-particle scattering, and the two-particle scattering is a mere exchange of quantum numbers. The state with the l interacting waves writes as

$$|k_1, k_2, \ldots k_l\rangle = \sum_{1 \le n_1 < \ldots < n_l \le L} a_{n_1,n_2,\ldots n_l}(k_1, k_2, \ldots k_l) \, |n_1, n_2, \ldots n_l\rangle \,. \tag{3.8}$$

The corresponding coefficients

$$a_{n_1,n_2,\ldots n_l}(k_1, k_2, \ldots k_l) = \sum_{\sigma \in \mathcal{S}_l} S(\sigma, k) \, \exp i[k_{\sigma(1)} n_1 + \cdots + k_{\sigma(l)} n_l] \,. \tag{3.9}$$

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Here $\mathcal{S}_l$ is the group of permutations, and the phase factor $S(\sigma, k)$ obeys the group property

$$S(\sigma_1\sigma_2, k) = S(\sigma_2, k)\, S(\sigma_1, \sigma_2 k)\,. \tag{3.10}$$

For the interchange of the two neighboring excitations $\sigma_{i,i+1}$ the phase factor

$$S(\sigma_{i,i+1}, k) \;=\; S(k_i, k_{i+1}) \;=\; -\frac{\mathrm{e}^{i(k_i+k_{i+1})} + 1 - 2\mathrm{e}^{ik_{i+1}}}{\mathrm{e}^{i(k_i+k_{i+1})} + 1 - 2\mathrm{e}^{ik_i}} \tag{3.11}$$

reduces to the two-particle scattering phase. Then the periodicity condition reads

$$\mathrm{e}^{ik_1 L} \prod_{j\neq 1} S(k_1, k_j) \;=\; 1\,. \tag{3.12}$$

The set of equations (3.12) is known as the Bethe ansatz equations (BAE).

These equations get modified for the orbifold gauge theory. After the diagonalization the action of the twist field $\gamma(g)$ on the fields Z and W can be brought to the form

$$g : \begin{pmatrix} Z \\ W \end{pmatrix} \;\to\; \begin{pmatrix} \omega^{s_Z} & 0 \\ 0 & \omega^{s_W} \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix}, \tag{3.13}$$

where $\omega = \mathrm{e}^{2\pi i/S}$; S being the order of the element g, $g^S = 1$. As it was argued, interaction terms are unaffected by the orbifoldization procedure except for the interaction between the first and the L-th site. As it was emphasized in [36], one can use the same bulk solution (3.8), though the periodicity condition as well as the zero momentum constraint will both acquire an extra phase factor. The simplest way to find these phases is to consider the plane wave solution.

Bethe Ansatz Equations can be generalized to the chains with an arbitrary underlying symmetry (super)algebra [37–41]. It is convenient to use the *rapidities* λ to describe the excitations. There exist the r types of excitations, corresponding to the r simple roots. Since there can be multiple excitations of the same type it is convenient to number the corresponding spectral parameters as $\lambda_{j,k}$; where $j = 1, 2, \ldots r$ and $k = 1, 2, \ldots K_j$, K_j being the number of excitations of type j. The set of the BAE becomes

$$\mathrm{e}^{iP_{j,k}L} \;=\; \prod_{(j',k')\neq(j,k)} S_{jj'}(\lambda_{j,k}, \lambda_{j',k'})\,; \tag{3.14}$$

where the scattering matrix and momenta are given by

$$S_{jj'} \;=\; \frac{\lambda_{j,k} - \lambda_{j',k'} + \frac{i}{2}a_{j,j'}}{\lambda_{j,k} - \lambda_{j',k'} - \frac{i}{2}a_{j,j'}}\,, \quad \mathrm{e}^{iP_{j,k}} \;=\; \frac{\lambda_{j,k} + \frac{i}{2}V_j}{\lambda_{j,k} - \frac{i}{2}V_j}\,. \tag{3.15}$$

(Here V_j are the Dynkin labels of the representation via which the algebra acts on each site — twice the spin in the $\mathfrak{su}_2$ case; and $a_{jj'}$ are the elements of the Cartan matrix.) The total energy of the corresponding eigenstate is

$$\epsilon \;=\; \sum_{j=1}^{r}\sum_{k=1}^{K_j} \epsilon_j(\lambda_{j,k})\,, \quad \epsilon_j(\lambda_{j,k}) \;=\; \frac{V_j}{\lambda_{j,k}^2 + \frac{1}{4}V_j^2}\,. \tag{3.16}$$

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The algebra behind the $\mathcal{N}=4$ supersymmetry is the $\mathfrak{su}_{2,2|4}$ superalgebra. Thus generic operators of the field theory get identified with some states of the $\mathfrak{su}_{2,2|4}$-symmetric spin chain. The whole $\mathcal{N}=4$ theory was proved to be integrable in [14–16]. The energy eigenvalues $E_{\mathcal{O}}$ of $\mathcal{H}$ are related to the anomalous dimensions $\Delta_{\mathcal{O}}$ of local single trace operators $\mathcal{O}$ by

$$\Delta_{\mathcal{O}} = \lambda\, E_{\mathcal{O}}, \tag{3.17}$$

with $\lambda = \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}$ the 't Hooft coupling.

Note that the spin chains with the different representations of $\mathfrak{su}_{2,2|4}$ correspond to different subclasses of operators in field theory. The two bosonic subalgebrae $\mathfrak{su}_{2,2}$ and $\mathfrak{su}_4$ of $\mathfrak{su}_{2,2|4}$ are nothing but the algebra of the conformal group in four dimensions and the $\mathcal{R}$-symmetry algebra. Unlike the bosonic semisimple Lie algebrae, the Dynkin diagram of a superalgebra is not unique. For the $\mathfrak{su}_{2,2|4}$ there exist the two distinguished choices of the root system, the so-called "Beauty" and the "Beast"; and they are discussed in [42]. Though the "Beast" is the most obvious system with one fermionic root, the "Beauty" root system proves useful in the context of $\mathcal{N}=4$ supersymmetry.

3.2 General orbifolds

As it was argued, there is no mixing between the different twisted sectors. Furthermore, in each given twisted sector $[g]$ one can construct all the states inserting one twist field $\gamma(g)$, g being any fixed representative of the conjugacy class $[g]$. In conjunction with the fact that one can diagonalize the action of any given element $g \in \Gamma$ — the problem reduces to the Abelian case modulo some subtleties. In particular, the S_g-invariance does not completely incorporate into Bethe equations; and it is to be imposed by hand — that is why some of the Bethe eigenstates may be projected out.

Therefore, one can apply techniques similar to those used in [26] for the study of Abelian orbifolds. Then each given element $g \in \Gamma \subset \mathrm{SU}_4$ can be brought to the diagonal form so that in SU(4) it becomes

$$\mathfrak{R}(g) = \begin{pmatrix} \mathrm{e}^{-2\pi i t_1/S} & & & 0 \\ & \mathrm{e}^{2\pi i (t_1-t_2)/S} & & \\ & & \mathrm{e}^{2\pi i (t_2-t_3)/S} & \\ 0 & & & \mathrm{e}^{2\pi i t_3/S} \end{pmatrix}. \tag{3.18}$$

Here S is the order of the element g, i.e., $g^S = 1$. For supersymmetric orbifolds that we consider the group Γ embeds into SU(3), and this imposes the extra restrictions on the weights t_i. Even though we need only the two independent parameters in order to describe embedding $\Gamma \subset \mathrm{SU}(3)$, it may be convenient to keep all the three parameters t_1, t_2 and t_3 in the calculations. In particular, it may account for different embeddings $\mathrm{SU}(3) \subset \mathrm{SU}(4)$ or different choices of the vacuum state.

The Bethe equations for the complete $\mathfrak{su}_{2,2|4}$ algebra acquire some extra phases:

$$\left(\frac{\lambda_{j,k} + \frac{i}{2}V_j}{\lambda_{j,k} - \frac{i}{2}V_j}\right)^L = \mathfrak{R}_j(g) \prod_{(j'.k')\neq(j,k)} \frac{\lambda_{j,k} - \lambda_{j',k'} + \frac{i}{2}a_{j,j'}}{\lambda_{j,k} - \lambda_{j',k'} - \frac{i}{2}a_{j,j'}}\,. \tag{3.19}$$

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Similarly, the momentum constraint reads

$$\mathfrak{R}_0(g) \prod_{j=1}^{7} \prod_{k=0}^{K_j} \frac{\lambda_{j,k} + \frac{i}{2} V_j}{\lambda_{j,k} - \frac{i}{2} V_j} = 1\,. \tag{3.20}$$

The phase factors

$$\mathfrak{R}_j(g) = \mathrm{e}^{2\pi i q_j / S}\,, \tag{3.21}$$

where the integers q_j depend on the choice of the root system:

$$\text{“Beauty”}: \overset{-t_2}{\odot}\quad \overset{0}{\ominus}\!\!-\!\!\!-\!\!\!-\overset{-t_1}{\otimes}\!\!-\!\!\!-\!\!\!-\overset{2t_1-t_2}{\oplus}\!\!-\!\!\!-\!\!\!-\overset{2t_2-t_1-t_3}{\oplus}\!\!-\!\!\!-\!\!\!-\overset{2t_3-t_2}{\oplus}\!\!-\!\!\!-\!\!\!-\overset{t_3}{\otimes}\!\!-\!\!\!-\!\!\!-\overset{0}{\ominus} \tag{3.22}$$

$$\text{“Beast”}: \overset{0}{\odot}\quad \overset{0}{\oplus}\!\!-\!\!\!-\!\!\!-\overset{0}{\oplus}\!\!-\!\!\!-\!\!\!-\overset{0}{\oplus}\!\!-\!\!\!-\!\!\!-\overset{t_1}{\otimes}\!\!-\!\!\!-\!\!\!-\overset{t_2-2t_1}{\ominus}\!\!-\!\!\!-\!\!\!-\overset{t_1-2t_2+t_3}{\ominus}\!\!-\!\!\!-\!\!\!-\overset{t_2-2t_3}{\ominus} \tag{3.23}$$

(The leftmost "root" corresponds to the phase $\mathfrak{R}_0(g) = \mathrm{e}^{2\pi i q_0/S}$.) Let us stress that this structure is the direct generalization of that in the $\mathfrak{su}_2$ subsector: the bulk ansatz remains unaltered, while the boundary conditions get modified. Recall that in the $\mathfrak{su}_2$ case there is a single root $\gamma_1 = \alpha_{12}$, and the weight $q_1 = s_W - s_Z \equiv s_2 - s_1$. Analogously, for an excitation associated with some simple root $\gamma = \alpha_{ij}$ the corresponding weight $q_\gamma = s_j - s_i$ is the difference of the two corresponding charges. The number q_0 is determined by the choice of the Bethe vacuum.

There is an elegant way to summarize all the Bethe equation and momentum constraint together. In order to do this one introduces the two new types of excitations to the existing seven types ($j = 1, \ldots, \mathrm{rk}\,\mathfrak{su}_{2,2|4} = 7$). The quasi-excitation of type $j = 0$ corresponds to the insertion of a new spin chain site. In order to have a length L chain one is to insert exactly the $K_0 = L$ excitations of type 0. The quasi-excitation of type $j = -1$ corresponds to the twist field. The scattering phases of the excitations are[6]

$$S_{j,j'} = \frac{\lambda_{j,k} - \lambda_{j',k'} + \frac{i}{2} a_{j,j'}}{\lambda_{j,k} - \lambda_{j',k'} - \frac{i}{2} a_{j,j'}}\,, \tag{3.24}$$

$$S_{j,0} = \frac{\lambda_{j,k} - \frac{i}{2} V_j}{\lambda_{j,k} + \frac{i}{2} V_j}\,, \quad S_{j,-1} = \mathfrak{R}_j(g)\,; \tag{3.25}$$

$$S_{0,0} = 1\,, \quad S_{0,-1} = \mathfrak{R}_0(g)\,. \tag{3.26}$$

Type 0 excitation do not have an associated spectral parameter, while type -1 excitations can have different twist classes $[g]$. Excitations of both type 0 and -1 do not contribute to the total energy.

With these notations we can therefore summarize all the Bethe equations as

$$\prod_{j'=-1}^{J} \prod_{\substack{k'=1 \\ (j',k') \neq (j,k)}}^{K_{j'}} S_{j,j'}(\lambda_{j,k}, \lambda_{j',k'}) = 1\,. \tag{3.27}$$

[6]Note that the scattering phase $S_{-1,-1}$ is not needed as we restrict ourselves to one excitation of type -1. Even though one may introduce several such excitations it would cause some unnecessary technical difficulties. As it was shown, insertion of a single twist field suffices to produce all the orbifold states.

The equations for $j = 1, \ldots, 7$ give the BAE (3.19), equation for $j = 0$ gives the momentum constraint (3.20),[7] and equation for $j = -1$ gives the "zero charge condition"

$$\mathfrak{R}_0(g)^L \prod_{j'=1}^{7} \mathfrak{R}_{j'}(g)^{K_{j'}} . \tag{3.28}$$

It implies the g-invariance of the corresponding state in the field theory. Again, let us stress that for a generic orbifold this condition is not sufficient, and there should be imposed a more restrictive invariance condition w.r.t. the full stabilizer S_g. As a result, some of the Bethe eigenstates may be projected out in field theory.

4. Example quivers

Here we study application of the Bethe equations to the two example quivers, ones with both Abelian and non-Abelian orbifold group. For these simple examples one can easily determine the anomalous dimensions of operators in the twisted sectors. Then these operators can be recast into the quiver notation. Generally operators corresponding to the closed paths in the quiver are *not* the eigenvectors of the matrix of anomalous dimensions. In other words, an operator corresponding to a closed loop in the quiver is typically a mix of operators with different conformal dimensions; neither does it belong to a given twisted sector.

4.1 Abelian $\mathbb{Z}_6$ quiver

Here we consider a simple example, $\mathbb{Z}_6$ quiver (see figure 4). We restrict ourselves to the $\mathfrak{su}_2$ subsector formed by the two scalars, Z with charge $s_Z = 1$ and W with charge $s_W = -2$. We will study the twisted sector with the twist γ^n, $n = 0, \ldots, 5$; γ being the generating element of $\mathbb{Z}_6$. Let us choose the length of the spin chain $L = 3$; then the vacuum can be chosen as $\mathrm{Tr}\left[\gamma ZZZ\right]$ — note that it will be projected out. There also exist the excited states with one or three W's, while the states with the two excitations will also be projected out.

Our goal will be to describe these Bethe vectors in terms of the quiver notation. By $\mathcal{O}_{ijki} \equiv \mathrm{Tr}\, \Phi^i_j \Phi^j_k \Phi^k_i$ we will denote the quiver gauge theory operator corresponding to the closed cycle between the three nodes $k \to j \to i$ in the quiver.

Note that the state with the three excitations is unique in each given twisted sector, and it corresponds to the field theory operator $\mathrm{Tr}\left[\gamma^n WWW\right]$. Commuting the twist field γ^n with one of the fields W we find that

$$\mathrm{Tr}\left[\gamma^n WWW\right] = \mathrm{e}^{2\pi i n s_W/6}\, \mathrm{Tr}\left[\gamma^n WWW\right] ; \tag{4.1}$$

i.e., this state is projected out in all sectors except for $n = 0$ and $n = 3$. The reason for this is the extra symmetry: it is sufficient to commute the twist field with only one of the

[7]Although there are L quasi-excitations of type 0, there is only one corresponding Bethe equation, because all of these quasi-excitations are equivalent, and they have no spectral parameter which might distinguish them.

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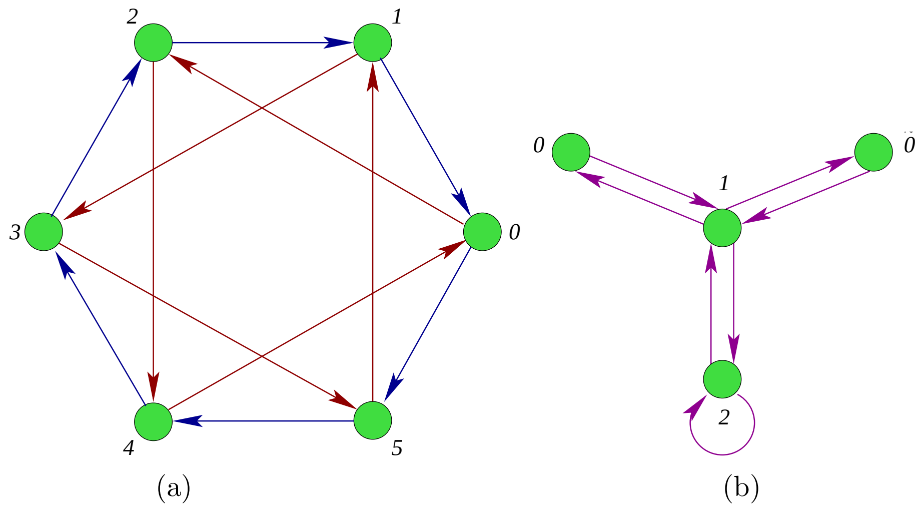

Figure 4: (a) The $\mathbb{Z}_6$ quiver. There are the six nodes corresponding to the six representations of $\mathbb{Z}_6$. We show only the scalar lines corresponding to the fields Z (blue lines) transforming in $\mathfrak{R}_Z \simeq \rho_1$ and W (red lines) transforming in $\mathfrak{R}_W \simeq \rho_4 \simeq \rho_{-2}$. (b) The D_5 quiver with the two-dimensional defining representation $\mathfrak{R} \simeq \rho_1$. Note that for these two quivers we show only the lines corresponding to the $\mathfrak{su}_2$ subsector; i.e., the two scalar fields.

three W fields. Note that the total charge of the three fields W is zero, and normally one would expect $\mathrm{Tr}\left[\gamma WWW\right]$ to be a non-trivial operator.

Using the formula (2.23) we find that

$$\mathcal{O}_{0420} = \mathrm{Tr}\left[WWW\right] + \mathrm{Tr}\left[\gamma^3 WWW\right], \tag{4.2}$$

$$\mathcal{O}_{1531} = \mathrm{Tr}\left[WWW\right] - \mathrm{Tr}\left[\gamma^3 WWW\right]; \tag{4.3}$$

or

$$\mathrm{Tr}\left[WWW\right] = \frac{1}{2}\mathcal{O}_{0420} + \frac{1}{2}\mathcal{O}_{1531}\,, \tag{4.4}$$

$$\mathrm{Tr}\left[\gamma^3 WWW\right] = \frac{1}{2}\mathcal{O}_{0420} - \frac{1}{2}\mathcal{O}_{1531}\,. \tag{4.5}$$

Graphically the operators $\mathcal{O}_{042}$ and $\mathcal{O}_{153}$ correspond to the two closed triangles formed by the red lines. Applying the Hamiltonian we find the anomalous dimensions

$$\Delta_{\mathrm{Tr}\left[\gamma^3 WWW\right]} = \Delta_{\mathrm{Tr}\left[\gamma^3 WWW\right]} = 0\,. \tag{4.6}$$

The states with one excitation have the form $\mathrm{Tr}\left[\gamma^n ZZW\right]$, and there is one such state in each given twisted sector. These operators correspond to the triangles with the two blue (field Z) and one red (field W) line. There are six such triangles and there are six different operators with $n = 1, \ldots, 5$ — these numbers coincide as we expect. The transition formula between these two descriptions is

$$\mathcal{O}_{l,\,l+1,\,l+2,\,l} = \sum_{n=0}^{5} \mathrm{e}^{-2\pi i \frac{ln}{6}}\,\mathrm{Tr}\left[\gamma^n ZZW\right]; \tag{4.7}$$

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performing the Fourier transform yields

$$\mathrm{Tr}\left[\gamma^n ZZW\right] = \frac{1}{6}\sum_{l=0}^{5} \mathrm{e}^{2\pi i \frac{ln}{6}} \mathcal{O}_{l,\,l+1,\,l+2,\,l}\,. \tag{4.8}$$

These operators diagonalize the matrix of anomalous dimensions. Direct application of the Hamiltonian shows that the corresponding eigenvalues are

$$\Delta_{\mathrm{Tr}\,[\gamma^n ZZW]} = 4\lambda \sin^2\frac{\pi n}{6}\,. \tag{4.9}$$

This simple example illustrates the interrelation of the two descriptions in the orbifold gauge theory. First, the quiver description gives a very clear understanding of what the physical fields and gauge invariant operators are, while in the "orbit" description using the twist fields some of the operators may be projected out. On the other hand, the description using the twist fields proves to be more robust for studying the field theory dynamics (the matrix of anomalous dimensions). In order to illustrate this let us write the part of the action responsible for the non-trivial part of the interaction Hamiltonian, $\mathrm{Tr}\left[ZWZ^\dagger W^\dagger + WZW^\dagger Z^\dagger\right]$. In terms of the quiver notation

$$\begin{aligned}\mathrm{Tr}\left[ZWZ^\dagger W^\dagger + WZW^\dagger Z^\dagger\right] &= \sum_l \left[\mathcal{O}_{l,l+1,l-1,l-2,l} + \mathcal{O}_{l,l-2,l-1,l+1,l}\right] \\ &= \sum_l \mathrm{Tr}\left[Z^l_{l+1}\, W^{l+1}_{l-1}\, Z^{l-2\,\dagger}_{l-1}\, W^{l-2\,\dagger}_{l} + W^l_{l-2}\, Z^{l-2}_{l-1}\, W^{l+1\,\dagger}_{l-1}\, Z^{l\,\dagger}_{l+1}\right].\end{aligned} \tag{4.10}$$

Here Z^k_l denotes the field corresponding to the quiver arrow going from node l to node k. Note that the conjugation changes the direction of the corresponding arrow; e.g., $Z^{1\,\dagger}_2$ is an arrow going from node 1 to node 2. Indeed, as we see, studying the matrix of anomalous dimensions using the quiver notation would have been more complicated.

4.2 Non-abelian D_5 quiver

Next we consider a simple orbifold with a non-Abelian discrete group D_5 (the facts about the dihedral group D_S as well as its representation ring are given in appendix B.) The corresponding quiver is shown in figure 4. Again, we study the $\mathfrak{su}_2$ sector, and the scalar field Φ^I transforms in the two-dimensional representation $\mathfrak{R} \simeq \rho_1$. From the quiver representation it is clear that there are the four different operators of length $L = 2$; namely, those are

$$\mathcal{O}_{010}\,, \quad \mathcal{O}_{\tilde{0}1\tilde{0}}\,, \quad \mathcal{O}_{121}\,, \quad \mathcal{O}_{222}\,. \tag{4.11}$$

On the other hand, there are the four different twist classes, $\{[e], [r], [r^2], [\sigma]\}$. Applying the definitions of the operators (A.28), we see that in each twist class there is exactly one non-trivial operator; thus there are the total of four operators of length two:

$$\mathcal{O}_e = \mathrm{Tr}\left[ZW\right], \;\; \mathcal{O}_r = \mathrm{Tr}\left[\gamma(r)ZW\right], \;\; \mathcal{O}_{r^2} = \mathrm{Tr}\left[\gamma(r^2)ZW\right], \;\; \mathcal{O}_\sigma = \mathrm{Tr}\left[\gamma(\sigma)ZZ\right]. \tag{4.12}$$

Here Z and W denote the first and second components of the field Φ^I. Note that the product of the two fields ZZ has transforms non-trivially under the action of r; nevertheless, in the sector with twist $[\sigma]$ the operator $\mathcal{O}_\sigma = \mathrm{Tr}\left[\gamma(\sigma)ZZ\right]$ is non-trivial as $r \notin S_\sigma$. The absence of mixing between the different twist classes ensures that the set of operators $\{\mathcal{O}_e,\ \mathcal{O}_r,\ \mathcal{O}_{r^2},\ \mathcal{O}_\sigma\}$ diagonalize the matrix of anomalous dimensions. Acting with the Hamiltonian we find the corresponding anomalous dimensions as

$$\Delta_{\mathcal{O}_e} = 0\,, \quad \Delta_{\mathcal{O}_r} = 4\lambda \sin^2\frac{\pi}{5} = \frac{5-\sqrt{5}}{2}\lambda\,, \quad \Delta_{\mathcal{O}_{r^2}} = 4\lambda \sin^2\frac{2\pi}{5} = \frac{5+\sqrt{5}}{2}\lambda\,, \quad \Delta_{\mathcal{O}_\sigma} = 0\,. \tag{4.13}$$

The same eigenvalues can be obtained solving the Bethe equations. The three operators $\mathcal{O}_e$, $\mathcal{O}_r$ and $\mathcal{O}_{r^2}$ are the states with one excitation. Diagonalizing the twist field as

$$\gamma(g) = \begin{pmatrix} \mathrm{e}^{i\alpha} & 0 \\ 0 & \mathrm{e}^{-i\alpha} \end{pmatrix}, \qquad g = e, r, r^2\,; \tag{4.14}$$

we find that the Bethe equation and the momentum constraint reduce to

$$\frac{\lambda + \frac{i}{2}}{\lambda - \frac{i}{2}} = \mathrm{e}^{i\alpha}\,, \qquad \epsilon = \frac{1}{\lambda^2 + \frac{1}{4}}\,. \tag{4.15}$$

This gives

$$\lambda = \frac{1}{2}\cot\frac{\alpha}{2}\,, \qquad \epsilon = 4\sinh^2\frac{\alpha}{2}\,. \tag{4.16}$$

For the twist element $g = e,\ r,\ r^2$ we have $\alpha = 0,\ 2\pi/5,\ 4\pi/5$ correspondingly. This reproduces the correct result. The twist field $\gamma(g)$ is non-diagonal. After the diagonalization of $\gamma(\sigma)$ operator $\mathcal{O}_\sigma$ maps to the vacuum state, and that is why $\Delta_{\mathcal{O}_\sigma} = 0$.

The next step is to find the dictionary between the two notations. In order to do this one can start with the quiver notation and rewrite the corresponding operators using the transition rules (A.21) and (A.22) from appendix A. The two operators $\mathcal{O}_{010}$ and $\mathcal{O}_{\tilde{0}1\tilde{0}}$ correspond to the closed paths $\rho_0 \leftarrow \rho_1 \leftarrow \rho_0$ and $\rho_{\tilde{0}} \leftarrow \rho_1 \leftarrow \rho_{\tilde{0}}$. Since the representations ρ_0 and $\rho_{\tilde{0}}$ are one-dimensional, the corresponding invariant tensors

$$\mathcal{K}_{AB}{}_1^1 = \mathcal{K}_{AB} \tag{4.17}$$

(the indices A, B belong to the defining representation $\mathfrak{R} \simeq \rho_1$.) The non-zero components are

$$\mathcal{K}_{12} = \mathcal{K}_{21} = \frac{1}{\sqrt{2}} \tag{4.18}$$

(note that the normalization respects the unitarity condition.) Then

$$\mathcal{K}_{AB}(g) = \mathcal{K}_{AB}\,\overline{\rho_\lambda}(g)\,, \qquad \lambda = 0, \tilde{0}\,. \tag{4.19}$$

This gives

$$\begin{aligned} \mathcal{O}_{010} &= \sqrt{2}\,\mathrm{Tr}\Big[ZW + (1+\omega)\gamma(r)ZW + (1+\omega^2)\gamma(r^2)ZW + 5\gamma(\sigma)ZZ\Big] \\ &= \sqrt{2}\Big[\mathcal{O}_e + (1+\omega)\mathcal{O}_r + (1+\omega^2)\mathcal{O}_{r^2} + 5\mathcal{O}_\sigma\Big]\,; \end{aligned} \tag{4.20}$$

$$\begin{aligned} \mathcal{O}_{\tilde{0}1\tilde{0}} &= \sqrt{2}\,\mathrm{Tr}\Big[ZW + (1+\omega)\gamma(r)ZW + (1+\omega^2)\gamma(r^2)ZW - 5\gamma(\sigma)ZZ\Big] \\ &= \sqrt{2}\Big[\mathcal{O}_e + (1+\omega)\mathcal{O}_r + (1+\omega^2)\mathcal{O}_{r^2} - 5\mathcal{O}_\sigma\Big]\,. \end{aligned} \tag{4.21}$$

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(We have used the permutation relation (A.31).)

Next, $\mathcal{O}_{121}$ corresponds to the product of the two tensors,

$$\mathcal{K}_{AB}{}_l^k = \sum_{p\in\rho_2} \mathcal{K}^p_{Al}\,\mathcal{K}^k_{Bp}\,, \qquad k,l \in \rho_1\,. \tag{4.22}$$

The non-trivial coefficients corresponding to the decomposition $\mathfrak{R}\otimes\rho_1 \to \rho_2$ are $\mathcal{K}^1_{11} = \mathcal{K}^2_{22} = 1$, while those corresponding to the decomposition $\mathfrak{R}\otimes\rho_2 \to \rho_1$ are $\mathcal{K}^1_{21} = \mathcal{K}^2_{12} = 1$. This gives the corresponding invariant tensor in (4.22):

$$\mathcal{K}_{12}{}_1^1 = \mathcal{K}_{21}{}_2^2 = 1\,. \tag{4.23}$$

Therefore, one identifies

$$\mathcal{O}_{121} = 2\big[\mathcal{O}_e + (\omega^2+\omega^4)\mathcal{O}_r + (\omega^3+\omega^4)\mathcal{O}_{r^2}\big]\,. \tag{4.24}$$

Similarly, for the operator $\mathcal{O}_{222}$ we need to find the decomposition $\mathfrak{R}\otimes\rho_2 \to \rho_2$. The non-trivial coefficients are $\mathcal{K}^2_{11} = \mathcal{K}^1_{22} = 1$. Consequently,

$$\mathcal{K}_{12}{}_1^1 = \mathcal{K}_{21}{}_2^2 = 1 \tag{4.25}$$

and

$$\mathcal{O}_{222} = 2\big[\mathcal{O}_e + 2\omega^3\mathcal{O}_r + (1+\omega)\mathcal{O}_{r^2}\big]\,. \tag{4.26}$$

These formulae give the transition between the two bases in the operator space.

5. Concluding remarks

As we have seen, methods of studying the Abelian orbifold gauge theories can be extended to arbitrary non-Abelian setups with minor modifications. The key argument is that the diagonalization of the twist field in each given twisted sector allows one to apply the techniques used for the Abelian case. Indeed, in a given twisted sector $[g]$ the BAE reduce to those in the Abelian theory; though some of the states may still be projected out. As a general rule, which states survive the projection is determined by the invariant tensors of the stabilizer subgroup S_g; although there can be present extra symmetries projecting out some of the conceivably non-trivial states. A useful feature is that there is no mixing between the different twisted sectors; and this superselection rule simplifies diagonalization of the matrix of anomalous dimensions. On the other hand, one can use the quiver gauge theory notation. In this language the problems with some states being projected out do not appear, but the matrix of anomalous dimensions becomes more complicated.

Diagonalization of the twist field allows one to extend some of the results of the AdS/CFT correspondence to the general orbifolds. In particular, exactly as in the Abelian case, orbifoldization amounts to appearance of the fractional mode numbers, on both closed string and Bethe equations side [23]. Given the well established full one-loop agreement between the classical energies and anomalous dimensions as the functions of the mode numbers in [43–45], the correspondence holds for the arbitrary orbifold at one loop.

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It seems that the higher loop techniques described in [26, 46] apply to the non-Abelian case as well. This would open a possibility of applying the existing powerful integrability techniques to the quiver gauge theories with reduced supersymmetry. In particular, one would need to verify that the duality relations between the roots of different types are not violated (e.g., by some states being projected out). This will be done elsewhere.

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A. Quiver vs orbit description

We show that the two descriptions of the quiver gauge theory field content are equivalent and develop explicit transition formulae between them. First we introduce the basis in the field space of the parent $\mathcal{N}=4$ theory. As we saw in section 2, Chan-Paton indices of the fields transform in the regular representation of the orbifold group Γ. Namely, the fields belong to

$$V^{\perp} \otimes V_{\rm reg}^{\oplus N} \otimes \bar{V}_{\rm reg}^{\oplus N} \simeq V^{\perp} \otimes V_{\rm reg} \otimes \bar{V}_{\rm reg} \otimes \mathbb{C}^N \otimes \mathbb{C}^{*N} \tag{A.1}$$

$V^{\perp}$ being the representation corresponding to the transverse indices. We are going to use the two orthonormal bases in the group algebra $V_{\rm reg} \simeq \mathbb{C}[\Gamma]$,

$$\{\mathrm{e}_g = g\} \tag{A.2}$$

and

$$\{\mathrm{E}^{\lambda}_{mn} = \frac{1}{\sqrt{S_\lambda}} \sum_g \overline{\rho^{\lambda}_{nm}}(g)\, g\}\,, \quad S_\lambda = \frac{|\Gamma|}{\dim V_\lambda}\,. \tag{A.3}$$

The group acts on them according to

$$h\,:\,\mathrm{e}_g \to \mathrm{e}_{hg}\,, \tag{A.4}$$

$$h\,:\,\mathrm{E}^{\lambda}_{mn} \to \sum_k \overline{\rho^{\lambda}_{nk}}(h^{-1})\,\mathrm{E}^{\lambda}_{mk} = \sum_k \mathrm{E}^{\lambda}_{mk}\;\rho^{\lambda}_{kn}(h)\,. \tag{A.5}$$

The relation between these bases is

$$\mathrm{e}_g = \sum_\lambda \frac{1}{\sqrt{S_\lambda}} \sum_{mn} \rho^{\lambda}_{nm}(g)\,\mathrm{E}^{\lambda}_{mn}\,, \tag{A.6}$$

$$\mathrm{E}^{\lambda}_{mn} = \frac{1}{\sqrt{S_\lambda}} \sum_g \overline{\rho^{\lambda}_{nm}}(g)\,\mathrm{e}_g\,. \tag{A.7}$$

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The dual bases are introduced according to $e^*_g(e_h) = \delta_{gh}$ and $E^{\lambda*}_{kl}(E^\mu_{mn}) = \delta^{\lambda\mu}\,\delta_{km}\,\delta_{ln}$. Next we construct the two bases in the field space. These basis vectors are to label the invariant configurations in the field space, thus we need to find the invariant configurations

$$\left(V^\perp \otimes V_{\rm reg}^{\oplus N} \otimes \bar{V}_{\rm reg}^{\oplus N}\right)^\Gamma \simeq \left(V^\perp \otimes V_{\rm reg} \otimes \bar{V}_{\rm reg}\right)^\Gamma \otimes \mathbb{C}^N \otimes \mathbb{C}^{*N}\,. \tag{A.8}$$

In what follows we are going to drop the trivial $\mathbb{C}^N \otimes \mathbb{C}^{*N}$ factor

Let us start with the gauge field. The "orbit" basis

$$\mathbf{t}_g = \sum_h e_h \otimes e^*_{hg} \tag{A.9}$$

has a natural interpretation in terms of invariant combinations of strings stretching between image branes. Similarly, the product $\mathbf{t}_g \circ \mathbf{t}_h = \mathbf{t}_{gh}$ has a natural interpretation in terms of gluing the ends of open strings. Note that the hermitian conjugate $\mathbf{t}^\dagger_g = \mathbf{t}_{g^{-1}}$. In order to build the "quiver" basis we note that E^λ_{mn} do *not* transform in the first index (recall that each representation $\mathfrak{R}_\lambda$ enters $\mathfrak{R}_{\rm reg}$ with multiplicity equal to $N_\lambda = \dim V_\lambda$ — and this is the first index of E^λ_{mn} that numbers these copies). Therefore, the combination

$$\mathbf{T}^\lambda_{mn} = \sum_k E^\lambda_{mk} \otimes E^{\lambda*}_{nk} \tag{A.10}$$

is Γ-invariant. The multiplication rule is $\mathbf{T}^\lambda_{mn} \circ \mathbf{T}^\mu_{kl} = \delta^{\lambda\mu}\,\delta_{kn}\,\mathbf{T}^\lambda_{ml}$. Hermitian conjugate $\mathbf{T}^{\lambda\dagger}_{mn} = \mathbf{T}^\lambda_{nm}$. Here we recognize the matrix algebra $\bigoplus_\lambda \mathfrak{gl}(v_\lambda) \simeq \mathbb{C}[\Gamma]$. Thus, in these two calculations we get the same answer; i.e., the algebrae of $\mathbf{t}_g$ and $\mathbf{T}^\lambda_{mn}$ are both isomorphic to the group algebra. A straightforward calculation shows that the two bases are related by a discrete Fourier transform,

$$\mathbf{t}_g = \sum_\lambda \sum_{km} \overline{\rho^\lambda_{mk}}(g)\,\mathbf{T}^\lambda_{mk}\,. \tag{A.11}$$

Now we can do a similar calculation for the scalar and spinor fields which have transverse indices with non-trivial transformation rules. In this case we have to find the invariant subspace $\left(V^\perp \otimes V_{\rm reg} \otimes \bar{V}_{\rm reg}\right)^\Gamma$. Denote the basis of the transverse representation $V^\perp$ as $\{f_A \equiv e_{\alpha,A}\}$. Then the "orbit" basis has the form

$$\mathbf{t}_{A,g} = \sum_h (h \triangleright f_A) \otimes e_h \otimes e^*_{hg}\,; \tag{A.12}$$

where in terms of components the action is $h \triangleright f_A = \sum_B \rho^\alpha_{BA}(h)\,f_B$. Since the representation $\mathfrak{R}_\alpha$ is real, hermitian conjugation acts according to $\mathbf{t}^\dagger_{A,g} = \sum_B \rho_{BA}(g^{-1})\mathbf{t}_{B,g^{-1}}$. To find the "quiver" basis we will need to find the invariant tensors in the product

$$\left(V^\perp \otimes V_{\rm reg} \otimes \bar{V}_{\rm reg}\right)^\Gamma \simeq \bigoplus_{\lambda,\mu} \left(V_\alpha \otimes V_\lambda \otimes V^*_\mu\right)^\Gamma \otimes \mathbb{C}^{N_\lambda} \otimes \mathbb{C}^{*N_\mu}\,.$$

To do it we decompose the product of representations $\mathfrak{R}_\alpha$ and $\mathfrak{R}_\lambda$ into a direct sum of irreducible representations. In particular, in terms of basis vectors

$$e_{\alpha A} \otimes e_{\lambda l} = \sum_{\mu m} \mathcal{K}^{\mu m}_{\alpha A,\lambda l}\,e_{\mu m} \quad \text{and} \quad e_{\mu m} = \sum_{A,l} \overline{\mathcal{K}^{\mu m}_{\alpha A,\lambda l}}\,e_{\alpha A} \otimes e_{\lambda l}\,.$$

Therefore, the invariant configuration is

$$\sum_m \mathrm{e}_{\mu m} \otimes \mathrm{e}^*_{\mu m} = \sum_{A,l,m} \overline{\mathcal{K}^{\mu m}_{\alpha A,\lambda l}}\, \mathrm{e}_{\alpha A} \otimes \mathrm{e}_{\lambda l} \otimes \mathrm{e}^*_{\mu m}\,.$$

(The field components of the invariant configuration are given by the invariant tensor, $\Phi^{Al}_m \sim \overline{\mathcal{K}^{\mu m}_{\alpha A,\lambda l}}$.) This gives

$$\mathbf{T}^{\lambda\mu}_{lm} = \sum_{A,i,j} \overline{\mathcal{K}^{\mu j}_{\alpha I,\lambda i}}\, \mathrm{f}_A \otimes \mathrm{E}^{\lambda}_{li} \otimes \mathrm{E}^{\mu *}_{mj}\,. \tag{A.13}$$

Note that here the indices l and m are the indices of the gauge groups at the corresponding nodes $\mathfrak{R}_\lambda$ and $\mathfrak{R}_\mu$. These indices appear owing to the fact that each representation $\mathfrak{R}_\lambda$ enters the decomposition of the regular representation $\mathfrak{R}_{\mathrm{reg}}$ with multiplicity N_λ. A calculation similar to that for the gauge field gives

$$\mathbf{t}_{A,g} = \sum_{\lambda,l} \sum_{\mu,km} \frac{\dim V_\lambda}{\dim V_\mu}\, \mathcal{K}^{\mu k}_{A,\lambda l}\, \overline{\rho^{\mu}_{km}}(g)\, \mathbf{T}^{\lambda\mu}_{lm}\,. \tag{A.14}$$

The presence of the factor $\mathcal{K}^{\mu\cdot}_{A,\lambda\cdot}$ restricts the sum over (λ,μ) only to those pairs which are connected by a line in the quiver. Using (A.14), we can find the relation between the field components in the two notations,

$$\phi^{\lambda\mu}_{lm} = \sum_g \sum_k \frac{N_\lambda}{N_\mu}\, \mathcal{K}^{\mu k}_{A,\lambda l}\, \overline{\rho^{\mu}_{km}}(g)\, \phi^A_g\,. \tag{A.15}$$

The matrix product of the quiver gauge theory gives rise to the (modified) convolution in terms of the group algebra. Particularly, the product of the two fields ϕ and ψ with the transverse indices transforming in the representations $\mathfrak{R}_\alpha$ and $\mathfrak{R}_\beta$ is

$$\sum_m \phi^{\lambda\mu}_{lm}\, \psi^{\mu\nu}_{mn} = \frac{N_\lambda}{N_\nu} \sum_{g,h} \mathcal{K}^{\nu r}_{\alpha A,\beta B,\lambda l}\, \overline{\rho^{\nu}_{nr}}(h^{-1}g^{-1})\; \rho^{\beta}_{BC}(g)\, \phi^A_g\, \psi^C_h\,. \tag{A.16}$$

Here $\mathcal{K}^{\nu r}_{\alpha A,\beta B,\lambda l} = \sum_p \mathcal{K}^{\mu p}_{\alpha A,\lambda l}\, \mathcal{K}^{\nu r}_{\beta B,\mu p}$ is (one of) the invariant tensors corresponding to the decomposition $\mathfrak{R}_\alpha \otimes \mathfrak{R}_\beta \otimes \mathfrak{R}_\lambda \to \mathfrak{R}_\nu$. Note that (A.16) has the same structure as (A.15), the product $\phi \circ \psi$ having the defining representation $\mathfrak{R}_\alpha \otimes \mathfrak{R}_\beta$. The convolution rule is

$$(\phi \circ \psi)^{AB}_g = \sum_h \phi^A_h\; \rho_{BC}(h)\, \psi^C_{h^{-1}g}\,. \tag{A.17}$$

Both formulae (A.16) and (A.17) are also valid for the gauge fields which have no transverse indices (trivial representation). In this case some matrix elements and decomposition tensors become degenerate. Let us stress that the multiplication rule (A.17) naturally corresponds to the standard matrix multiplication in the parent $\mathrm{SU}(|\Gamma|\,N)$ theory. This means that

$$\sum_f \phi^A_{hf}\, \psi^B_{fg} = \rho^{\alpha}_{AA'}(h)\; \rho^{\beta}_{BB'}(h)\, (\phi \circ \psi)^{A'B'}_{h^{-1}g}\,. \tag{A.18}$$

This way of multiplication is induced from the original theory, and that is why it respects the gauge transformations. Another nice feature of the formula (A.16) is that when there exist several arrows going between different nodes the choice of a given arrow affects only the choice of the invariant tensors and does not affect the convolution product (A.17). It means that all the operators corresponding to the different paths (not necessarily closed) in the quiver formed by L consequent scalar lines $\lambda_1 \to \lambda_2 \to \cdots \to \lambda_{L+1}$ are contained in the product $\phi^{A_1} \ldots \phi^{A_L}$.

We can summarize these results as follows. An operator formed in the quiver notation as the product $\phi^{\lambda\nu_1} \phi^{\nu_1\nu_2} \ldots \phi^{\nu_{L-1}\mu}$ can be recast as

$$\Big(\phi \circ \cdots \circ \phi\Big)^{\lambda\mu}_{lm} = \sum_g \sum_k \frac{N_\lambda}{N_\mu} \mathcal{K}_{A_1 \ldots A_L}{}^{\mu k}_{\lambda l} \, \overline{\rho^{\mu}_{km}}(g) \Big(\phi \circ \cdots \circ \phi\Big)^{A_1 \ldots A_L}_g \; ; \qquad \text{(A.19)}$$

where the invariant tensor $\mathcal{K}$ is the one corresponding to the decomposition $\mathfrak{R}_\lambda \otimes \mathfrak{R}_{\nu_1} \otimes \cdots \otimes \mathfrak{R}_{\nu_{L-1}} \to \mathfrak{R}_\mu$. The product of fields in the r.h.s. is calculated according to (A.17). In its turn it is related to the product of the fields of the original $\mathcal{N} = 4$ theory as

$$\Big(\phi^{A_1}, \ldots \phi^{A_L}\Big)_{h,hg} = \sum_{B_1,\ldots,B_L} \rho_{A_1 B_1}(h) \ldots \rho_{A_L B_L}(h) \Big(\phi \circ \cdots \circ \phi\Big)^{B_1 \ldots B_L}_g . \qquad \text{(A.20)}$$

These formulae will be of crucial importance for constructing the gauge invariant observables.

A.1 Construction of observables

In order to construct gauge invariant observables it is convenient to use the quiver notation. Taking a closed loop in the quiver and using (A.19) one can write the corresponding operator as

$$\mathrm{Tr}_{\lambda_1} \phi^{\lambda_1\lambda_2} \phi^{\lambda_2\lambda_3} \ldots \phi^{\lambda_L\lambda_1} = \sum_g \sum_{k,l} \mathcal{K}_{A_1,\ldots A_L}{}^{\lambda_1 k}_{\lambda_1 l} \, \overline{\rho^{\lambda}_{kl}}(g) \Big(\phi \circ \ldots \circ \phi\Big)^{A_1,\ldots A_L}_g \; ; \quad \text{(A.21)}$$

where the invariant tensor

$$\mathcal{K}_{A_1,\ldots A_L}{}^{\lambda_1 k}_{\lambda_1 l} = \sum_{l_2,\ldots,l_L} \mathcal{K}^{\lambda_2 l_2}_{A_1\,\lambda_1 l} \mathcal{K}^{\lambda_3 l_3}_{A_2\,\lambda_2 l_2} \ldots \mathcal{K}^{\lambda_L l_L}_{A_{L-1}\,\lambda_{L-1} l_{L-1}} \mathcal{K}^{\lambda_1 k}_{A_L\,\lambda_L l_L} \qquad \text{(A.22)}$$

corresponds to the closed path $\lambda_1 \to \lambda_L \to \cdots \to \lambda_2 \to \lambda_1$. Note that the l.h.s. is explicitly symmetric w.r.t. the cyclic permutations of the fields under the trace. There also exists a different way to construct gauge invariant operators. Namely, let us start with the ansatz

$$\mathcal{O}[\mathcal{K}] = \sum_g \sum_{A_1,\ldots A_L} \mathcal{K}_{A_1 \ldots A_L}(g) \Big(\phi \circ \ldots \circ \phi\Big)^{A_1,\ldots A_k}_g . \qquad \text{(A.23)}$$

Generally such an expression represents a sum of operators corresponding to some paths in the quiver, not necessarily closed. That is why the gauge invariance condition has to be imposed separately, and it yields

$$\mathcal{K}_{B_1 \ldots B_L}(h^{-1}gh) = \sum_{A_1 \ldots A_L} \mathcal{K}_{A_1 \ldots A_L}(g)\, \rho_{A_1 B_1}(h) \ldots \rho_{A_L B_L}(h) . \qquad \text{(A.24)}$$

A straightforward consequence of this result is that $\mathcal{K}[g]$ has to be an invariant tensor w.r.t. the stabilizer subgroup S_g. Note that in (A.21) we had

$$\mathcal{K}_{A_1\ldots A_L}(g) = \sum_{k,l} \mathcal{K}_{A_1,\ldots A_L}{}^{\lambda k}_{\lambda l}\, \overline{\rho^{\lambda}_{kl}}(g)\,, \tag{A.25}$$

and it obviously satisfies (A.24). On the other side, tensor $\mathcal{K}(g)$ can be expanded in Fourier series as a function on the group,

$$\mathcal{K}_{A_1\ldots A_L}(g) = \sum_{\lambda}\sum_{k,l} \tilde{\mathcal{K}}^{(\lambda)}_{A_1,\ldots A_L}{}^{\lambda k}_{\lambda l}\, \overline{\rho^{\lambda}_{kl}}(g)\,; \tag{A.26}$$

and then the condition (A.24) translates into the requirement that the coefficients $\tilde{\mathcal{K}}^{(\lambda)}_{A_1,\ldots A_L}{}^{\lambda k}_{\lambda l}$ are invariant tensors. These considerations provide a dictionary between the two notations in the quiver gauge theory.

It is very important that the gauge invariance condition (A.24) relates the values of the tensor $\mathcal{K}(g)$ within the same conjugacy class, and there is no relation between the values of $\mathcal{K}$ on the different conjugacy classes. That is why one can build a gauge invariant operator with $\mathcal{K}(g) \neq 0$ only on a given conjugacy class $[g]$. Such operators are said to belong to the *twisted sector* with the twist $[g]$ (determined only up to a conjugation). One can choose a reference element g in the conjugacy class $[hgh^{-1}]$ and set $\mathcal{K}_{A_1\ldots A_L}(g) = \mathcal{K}_{A_1\ldots A_L}$, $\mathcal{K}_{A_1\ldots A_L}$ being some S_g-invariant tensor. Then (A.24) determines the values of $\mathcal{K}(h^{-1}gh)$ on all the elements of the conjugacy class. The corresponding operator is

$$\mathcal{O}[\mathcal{K}] = \sum_{g,h}\sum_{A_1\ldots A_L}\sum_{B_1\ldots B_L} \mathcal{K}_{A_1\ldots A_L}\, \rho_{A_1 B_1}(h)\ldots\, \rho_{A_L B_L}(h)\big(\phi\circ\cdots\phi\big)^{B_1\ldots B_L}_{h^{-1}gh}\;; \tag{A.27}$$

and it rewrites in terms of the fields of the parent $\mathcal{N}=4$ theory as

$$\mathcal{O}[\mathcal{K}] = \sum_{g,h}\sum_{A_1\ldots A_L} \mathcal{K}_{A_1\ldots A_L}\, \mathrm{Tr}\left[\gamma(g)\phi^{A_1}\ldots\phi^{A_L}\right]. \tag{A.28}$$

The twist field $\gamma(g)$ acts on the dynamical fields as follows,

$$\big(\phi^A\,\gamma(g)\big)_{h_1,h_2} = \phi^A_{h_1,gh_2}\,, \tag{A.29}$$

$$\big(\gamma(g)\,\phi^A\big)_{h_1,h_2} = \phi^A_{g^{-1}h_1,h_2}\,. \tag{A.30}$$

Invariance condition imposed by the orbifold projection on the fields implies the interchange relation

$$\big(\gamma(g)\,\phi^A\big) = \rho_{AB}(g^{-1})\,\big(\phi^B\,\gamma(g)\big)\,. \tag{A.31}$$

B. Representation ring of the dihedral group

The dihedral group D_S is generated by the two elements, r and σ, with the additional relations

$$r^S = \sigma^2 = 1\,, \qquad r\sigma = \sigma r^{-1}\,. \tag{B.1}$$

	$[e]$	$[r^m]$	$[\sigma]$
χ_0	1	1	1
$\chi_{\tilde{0}}$	1	1	-1
χ_l	2	$2\cos\left(2\pi\frac{lm}{S}\right)$	0

$\otimes$	ρ_0	$\rho_{\tilde{0}}$	ρ_k
ρ_0	ρ_0	$\rho_{\tilde{0}}$	ρ_k
$\rho_{\tilde{0}}$	$\rho_{\tilde{0}}$	ρ_0	ρ_k
ρ_l	ρ_l	ρ_l	$\begin{cases} \rho_{k+l} \oplus \rho_{k-l}\,, & k \neq l \\ \rho_{2l} \oplus \rho_0 \oplus \rho_{\tilde{0}}\,, & k = l \end{cases}$

g	e	r	r^2	$[\sigma]$
S_g	D_5	$\{e, r, \ldots, r^4\} \simeq \mathbb{Z}_5$	$\{e, r, \ldots, r^4\} \simeq \mathbb{Z}_5$	$\{e, \sigma\}$

Table 1: Table of characters and representation ring (multiplication table) of the dihedral group $D_{S=2n+1}$. Stabilizer subgroups S_g for a representative of each conjugacy class of the group D_5.

The order of the group $|D_S| = 2S$. We will restrict ourselves to the odd $S = 2n + 1$. Then there are the $n + 2$ conjugacy classes, $\mathcal{O}_1 = \{e\}$, $\mathcal{O}_2 = \{r, r^{2n}\}, \ldots$, $\mathcal{O}_{n+1} = \{r^n, r^{n+1}\}$, $\mathcal{O}_{n+2} = \{\sigma, \sigma r, \ldots, \sigma r^{2n}\}$. Thus there exist the $n + 2$ irreducible representations. Among them there are the n two-dimensional representations ρ_m:

$$\rho_m(r) = \begin{pmatrix} \omega^m & 0 \\ 0 & \omega^{-m} \end{pmatrix}, \qquad \rho_m(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad m = 1, 2, \ldots n\,. \tag{B.2}$$

Here $\omega = \mathrm{e}^{2\pi i/S}$ There are also the two one-dimensional representations ρ_0 and $\rho_{\tilde{0}}$:

$$\rho_0(r) = 1\,, \quad \rho_0(\sigma) = 1\,; \qquad \rho_{\tilde{0}}(r) = 1\,, \quad \rho_{\tilde{0}}(\sigma) = -1\,. \tag{B.3}$$

The table of characters as well as the representation ring and the stabilizer subgroups of each element of the group D_{2n+1} are shown in table 1.

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